ON CERTAIN THEOREMS OF JACOBI RELATING TO CANONICAL EQUATIONS.

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Jacobi in his Vorlesungen über Dynamik first proves the particular theorem contained in Art. 5 of the present paper, and afterwards derives therefrom that given in Art. 6. In what follows it will be seen that each of these is made a special case of the more general theorem given in Art. 1.

1. If we have given the partial differential equations

$$\frac{\partial S}{\partial t} + H = 0\,, \tag{a}$$

$$y_1 = \frac{\partial S}{\partial x_1}\,, \quad y_2 = \frac{\partial S}{\partial x_2}\,, \quad \ldots\,, \quad y_{2n} = \frac{\partial S}{\partial x_{2n}}\,, \tag{b}$$

in which H is a function of t and of the x's and y's, but S is a function of t and the x's only; and if there exist the so-called canonical relations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}\,, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}\,; \quad (i = 1, 2, \ldots, n) \tag{c}$$

then will similar canonical relations exist between the remaining variables; that is to say, we shall have

$$\frac{dx_{n+i}}{dt} = \frac{\partial H}{\partial y_{n+i}}\,, \quad \frac{dy_{n+i}}{dt} = -\frac{\partial H}{\partial x_{n+i}}\,. \quad (i = 1, 2, \ldots, n) \tag{d}$$

Differentiating each of the y's totally with reference to t, we have

$$\left.\begin{array}{l}
\dfrac{\partial y_1}{\partial t} + \sum_1^{2n} \dfrac{\partial y_1}{\partial x_i}\dfrac{dx_i}{dt} - \dfrac{dy_1}{dt} = 0\,, \\
\dfrac{\partial y_2}{\partial t} + \sum_1^{2n} \dfrac{\partial y_2}{\partial x_i}\dfrac{dx_i}{dt} - \dfrac{dy_2}{dt} = 0\,, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\dfrac{\partial y_{2n}}{\partial t} + \sum_1^{2n} \dfrac{\partial y_{2n}}{\partial x_i}\dfrac{dx_i}{dt} - \dfrac{dy_{2n}}{dt} = 0\,.
\end{array}\right\} \tag{1}$$

Also, since

$$\frac{\partial}{\partial x_i}\frac{\partial S}{\partial t} = \frac{\partial}{\partial t}\frac{\partial S}{\partial x_i} = \frac{\partial y_i}{\partial t}\,,$$

if we differentiate equations (a) partially with reference to $x_1, x_2, \ldots, x_{2n}$, we shall have

$$\left.\begin{aligned}
&\frac{\partial y_1}{\partial t} + \frac{\partial H}{\partial x_1} + \sum_1^{2n} \frac{\partial H}{\partial y_i}\frac{\partial y_i}{\partial x_1} = 0\,, \\
&\frac{\partial y_2}{\partial t} + \frac{\partial H}{\partial x_2} + \sum_1^{2n} \frac{\partial H}{\partial y_i}\frac{\partial y_i}{\partial x_2} = 0\,, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\frac{\partial y_{2n}}{\partial t} + \frac{\partial H}{\partial x_{2n}} + \sum_1^{2n} \frac{\partial H}{\partial y_i}\frac{\partial y_i}{\partial x_{2n}} = 0\,.
\end{aligned}\right\} \tag{2}$$

Since

$$\frac{\partial}{\partial x_j}\frac{\partial S}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial S}{\partial x_j}\,,$$

we may write

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_j}{\partial x_i} = a_{i,j}\,;$$

whence by reason of equations (c), we may also write

$$\frac{\partial y_1}{\partial t} + \sum_1^n \frac{\partial y_1}{\partial x_i}\frac{dx_i}{dt} - \frac{dy_1}{dt} = \frac{\partial y_1}{\partial t} + \frac{\partial H}{\partial x_1} + \sum_1^n \frac{\partial H}{\partial y_i}\frac{\partial y_i}{\partial x_1} = \lambda_1\,,$$

$$\frac{\partial y_2}{\partial t} + \sum_1^n \frac{\partial y_2}{\partial x_i}\frac{dx_i}{dt} - \frac{dy_2}{dt} = \frac{\partial y_2}{\partial t} + \frac{\partial H}{\partial x_2} + \sum_1^n \frac{\partial H}{\partial y_i}\frac{\partial y_i}{\partial x_2} = \lambda_2\,,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{\partial y_n}{\partial t} + \sum_1^n \frac{\partial y_n}{\partial x_i}\frac{dx_i}{dt} - \frac{dy_n}{dt} = \frac{\partial y_n}{\partial t} + \frac{\partial H}{\partial x_n} + \sum_1^n \frac{\partial H}{\partial y_i}\frac{\partial y_i}{\partial x_n} = \lambda_n\,.$$

$$\frac{\partial y_{n+1}}{\partial t} + \sum_1^n \frac{\partial y_{n+1}}{\partial x_i}\frac{dx_i}{dt} = \frac{\partial y_{n+1}}{\partial t} + \sum_1^n \frac{\partial H}{\partial y_i}\frac{\partial y_i}{\partial x_{n+1}} = \mu_1\,,$$

$$\frac{\partial y_{n+2}}{\partial t} + \sum_1^n \frac{\partial y_{n+2}}{\partial x_i}\frac{dx_i}{dt} = \frac{\partial y_{n+2}}{\partial t} + \sum_1^n \frac{\partial H}{\partial y_i}\frac{\partial y_i}{\partial x_{n+2}} = \mu_2\,,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{\partial y_{2n}}{\partial t} + \sum_1^n \frac{\partial y_{2n}}{\partial x_i}\frac{dx_i}{dt} = \frac{\partial y_{2n}}{\partial t} + \sum_1^n \frac{\partial H}{\partial y_i}\frac{\partial y_i}{\partial x_{2n}} = \mu_n\,.$$

Substituting λ_i, μ_i, $a_{i,j}$ for the corresponding quantities in equations (1) and (2), the latter become

$$
\left.
\begin{aligned}
\lambda_1 + \sum_1^n a_{1,\,n+i}\frac{dx_{n+i}}{dt} &= 0\,,\\
\lambda_2 + \sum_1^n a_{2,\,n+i}\frac{dx_{n+i}}{dt} &= 0\,,\\
\cdots\cdots\cdots&\cdots\\
\lambda_n + \sum_1^n a_{n,\,n+i}\frac{dx_{n+i}}{dt} &= 0\,,\\
\mu_1 - \frac{dy_{n+1}}{dt} + \sum_1^n a_{n+1,\,n+i}\frac{dx_{n+i}}{dt} &= 0\,,\\
\mu_2 - \frac{dy_{n+2}}{dt} + \sum_1^n a_{n+2,\,n+i}\frac{dx_{n+i}}{dt} &= 0\,,\\
\cdots\cdots\cdots&\cdots\\
\mu_n - \frac{dy_{2n}}{dt} + \sum_1^n a_{2n,\,n+i}\frac{dx_{n+i}}{dt} &= 0\,;
\end{aligned}
\right\}\qquad (3)
$$

$$
\left.
\begin{aligned}
\lambda_1 + \sum_1^n a_{1,\,n+i}\frac{\partial H}{\partial y_{n+i}} &= 0\,,\\
\lambda_2 + \sum_1^n a_{2,\,n+i}\frac{\partial H}{\partial y_{n+i}} &= 0\,,\\
\cdots\cdots\cdots&\cdots\\
\lambda_n + \sum_1^n a_{n,\,n+i}\frac{\partial H}{\partial y_{n+i}} &= 0\,,\\
\mu_1 + \frac{\partial H}{\partial x_{n+1}} + \sum_1^n a_{n+1,\,n+i}\frac{\partial H}{\partial y_{n+i}} &= 0\,,\\
\mu_2 + \frac{\partial H}{\partial x_{n+2}} + \sum_1^n a_{n+2,\,n+i}\frac{\partial H}{\partial y_{n+i}} &= 0\,,\\
\cdots\cdots\cdots&\cdots\\
\mu_n + \frac{\partial H}{\partial x_{2n}} + \sum_1^n a_{2n,\,n+i}\frac{\partial H}{\partial y_{n+i}} &= 0\,.
\end{aligned}
\right\}\qquad (4)
$$

If in (3) we consider dx_{n+i}/dt and $-\,dy_{n+i}/dt$ $(i = 1, 2, \ldots, n)$, and in (4) $\partial H/\partial x_{n+i}$ and $\partial H/\partial y_{n+i}$ $(i = 1, 2, \ldots, n)$, as the unknown quantities, we shall have two systems of $2n$ equations of the first degree involving $2n$ unknown quantities. In the two systems the coefficients and the known terms are the

same ; hence the corresponding unknown quantities must be the same : whence

$$\frac{dx_{n+i}}{dt} = \frac{\partial H}{\partial y_{n+i}}, \quad \frac{dy_{n+i}}{dt} = -\frac{\partial H}{\partial x_{n+i}}. \quad (i = 1, 2, \ldots, n) \tag{d}$$

Q. E. D.

2. It may be proved similarly that if

$$\frac{dx_{n+1}}{dt} + \frac{\partial H}{\partial y_{n+1}}, \quad \frac{dy_{n+i}}{dt} = -\frac{\partial H}{\partial x_{n+i}};$$

then

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i},$$

and, in general, if any $2n$ of the $4n$ relations (c) and (d) exist, the remaining $2n$ relations will also hold true.

3. If H be expressed in terms of t and the x's only, we shall have

$$\frac{dx_i}{dt} = 0; \quad (i = 1, 2, \ldots, 2n)$$

and, by integration,

$$x_i = g_i,$$

in which the g's are arbitrary constants ; whence

$$\frac{dy_i}{dt} = -\frac{\partial H}{\partial g_i}.$$

4. Similarly, if H be expressed in terms of t and the y's only, we shall have

$$\frac{dy_i}{dt} = 0; \; (i = 1, 2, \ldots, 2n)$$

and, by integration,

$$y_i = g_i',$$

in which the g''s are arbitrary constants ; whence

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial g_i'}.$$

5. If H be a function of t, x_1, x_2, ..., x_n, y_1, y_2, ..., y_n only, we shall have

$$\frac{dx_{n+i}}{dt} = 0, \quad \frac{dy_{n+i}}{dt} = 0; \quad (i = 1, 2, \ldots, n)$$

and, by integration,

$$x_{n+i} = h_i , \quad y_{n+i} = h_i', \quad (i = 1, 2, \ldots , n)$$

in which the h's and h''s are arbitrary constants. In this case S may be stated to be a function of t and the n variables x_i and containing n arbitrary constants h_i. The equations

$$y_i = \frac{\partial S}{\partial x_i}, \quad h_i' = \frac{\partial S}{\partial h_i} \quad (i = 1, 2, \ldots , n)$$

may then be defined as the general integrals of the simultaneous differential equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}. \quad (i = 1, 2, \ldots , n)$$

6. If

$$H = (H - R) + R ,$$

in which $H - R$ is a function of x_i's and y_i's only, while R is expressed in terms of x_{n+i}'s and y_{n+i}'s only; we shall have

$$\left.\begin{aligned} &\frac{\partial H}{\partial x_i} = \frac{\partial (H-R)}{\partial x_i}, \quad &\frac{\partial H}{\partial y_i} = \frac{\partial (H-R)}{\partial y_i}, \\ &\frac{\partial H}{\partial x_{n+i}} = \frac{\partial R}{\partial x_{n+i}}, \quad &\frac{\partial H}{\partial y_{n+i}} = \frac{\partial R}{\partial y_{n+i}}; \end{aligned}\right\} (i = 1, 2, \ldots , n)$$

whence, if

$$\frac{dx_i}{dt} = \frac{\partial (H-R)}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial (H-R)}{\partial x_i},$$

then

$$\frac{dx_{n+i}}{dt} = \frac{\partial R}{\partial y_{n+i}}, \quad \frac{dy_{n+i}}{dt} = -\frac{\partial R}{\partial x_{n+i}};$$

or, employing the idea of variation of constants,

$$\frac{dh_i}{dt} = \frac{\partial R}{\partial h_i'}, \quad \frac{dh_i'}{dt} = -\frac{\partial R}{\partial h_i}.$$

7. If we choose, the x's and y's may be divided into two groups containing $2n$ variables each, i. e. n x's and n y's, and by solving equations (b) the members of each group may be expressed as functions of the members of the other group only. Suppose such to be the case, and x_i, y_i to be any two members of one of the groups, and x_j, y_j to be any two members of the

remaining group. Then will

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_j}{\partial x_i}, \qquad \text{(a}'\text{)} \qquad\qquad \frac{\partial x_i}{\partial y_j} = \frac{\partial x_j}{\partial y_i}, \qquad \text{(c}'\text{)}$$

$$\frac{\partial y_i}{\partial y_j} = -\frac{\partial x_j}{\partial x_i} \qquad \text{(b}'\text{)} \qquad\qquad \frac{\partial x_i}{\partial x_j} = -\frac{\partial y_j}{\partial y_i}. \qquad \text{(d}'\text{)}$$

If i' and j' be particular values of i and j, we have from equations (b), remembering that variables belonging to the same group are independent of one another,

$$\frac{\partial y_{i'}}{\partial x_{j'}} = \frac{\partial}{\partial x_{j'}}\frac{\partial S}{\partial x_{i'}} + \Sigma'_i \left[\frac{\partial}{\partial x_i}\frac{\partial S}{\partial x_{i'}} \cdot \frac{\partial x_i}{\partial x_{j'}}\right], \tag{1'}$$

$$\frac{\partial y_i}{\partial x_{i'}} = \frac{\partial}{\partial x_{i'}}\frac{\partial S}{\partial x_i} + \Sigma'_j \left[\frac{\partial}{\partial x_j}\frac{\partial S}{\partial x_i} \cdot \frac{\partial x_j}{\partial x_{i'}}\right] = 0\,;$$

whence

$$\frac{\partial}{\partial x_i}\frac{\partial S}{\partial x_{i'}} = \frac{\partial}{\partial x_{i'}}\frac{\partial S}{\partial x_i} = -\Sigma'_j \left[\frac{\partial}{\partial x_j}\frac{\partial S}{\partial x_i} \cdot \frac{\partial x_j}{\partial x_{i'}}\right]. \tag{2'}$$

Substituting (2′) in (1′), we have

$$\frac{\partial y_{i'}}{\partial x_{j'}} = \frac{\partial}{\partial x_{j'}}\frac{\partial S}{\partial x_{i'}} - \Sigma'_i\,\Sigma'_j \left[\frac{\partial}{\partial x_j}\frac{\partial S}{\partial x_i} \cdot \frac{\partial x_j}{\partial x_{i'}} \cdot \frac{\partial x_i}{\partial x_{j'}}\right].$$

Similarly, interchanging i, i' and j, j',

$$\frac{\partial y_{j'}}{\partial x_{i'}} = \frac{\partial}{\partial x_{i'}}\frac{\partial S}{\partial x_{j'}} - \Sigma'_j\,\Sigma'_i \left[\frac{\partial}{\partial x_i}\frac{\partial S}{\partial x_j} \cdot \frac{\partial x_i}{\partial x_{j'}} \cdot \frac{\partial x_j}{\partial x_{i'}}\right];$$

whence

$$\frac{\partial y_{i'}}{\partial x_{j'}} = \frac{\partial y_{j'}}{\partial x_{i'}}; \quad \text{or, in general,} \quad \frac{\partial y_i}{\partial x_j} = \frac{\partial y_j}{\partial x_i}. \tag{a'}$$

As may easily be seen,

$$\frac{\partial y_i}{\partial x_i} = \Sigma'_j \left[\frac{\partial y_i}{\partial x_j}\frac{\partial x_j}{\partial x_i} + \frac{\partial y_i}{\partial y_j}\frac{\partial y_j}{\partial x_i}\right] = 0\,, \tag{3'}$$

$$\frac{\partial y_j}{\partial x_j} = \Sigma'_i \left[\frac{\partial y_j}{\partial x_i}\frac{\partial x_i}{\partial x_j} + \frac{\partial y_j}{\partial y_i}\frac{\partial y_i}{\partial x_j}\right] = 0\,, \tag{4'}$$

$$\frac{\partial x_i}{\partial y_i} = \Sigma'_j \left[\frac{\partial x_i}{\partial x_j}\frac{\partial x_j}{\partial y_i} + \frac{\partial x_i}{\partial y_j}\frac{\partial y_j}{\partial y_i}\right] = 0\,, \tag{5'}$$

$$\frac{\partial x_j}{\partial y_j} = \Sigma'_i \left[\frac{\partial x_j}{\partial x_i}\frac{\partial x_i}{\partial y_j} + \frac{\partial x_j}{\partial y_i}\frac{\partial y_i}{\partial y_j}\right] = 0\,. \tag{6'}$$

By reason of (a′), (3′) and (4′) may be written

$$\Sigma_j \frac{\partial y_i}{\partial x_j}\left[\frac{\partial x_j}{\partial x_i} + \frac{\partial y_i}{\partial y_j}\right] = 0\,,$$

$$\Sigma_i \frac{\partial y_i}{\partial x_j}\left[\frac{\partial x_i}{\partial x_j} + \frac{\partial y_j}{\partial y_i}\right] = 0\,;$$

whence (b′) and (d′) are derived directly by indeterminate coefficients. Also, by reason of (b′) and (d′), (5′) and (6′) may be written

$$\Sigma_j \frac{\partial x_i}{\partial x_j}\left[\frac{\partial x_j}{\partial y_i} - \frac{\partial x_i}{\partial y_j}\right] = 0\,,$$

$$\Sigma_i \frac{\partial x_j}{\partial x_i}\left[\frac{\partial x_j}{\partial y_i} - \frac{\partial x_i}{\partial y_j}\right] = 0\,;$$

from either of which (c′) follows immediately by indeterminate coefficients.

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